# Fourier Analysis

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## 1. Review.

For $r \in (0,1)$, define

$$P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}$$

Then $(P_r)_{r\in(0,1)}$ is called the Poisson kernel as $r \to 1$.

**Corollary:** Let $f$ be integrable on the circle.

Then

① $P_r * f(x) \longrightarrow f(x)$ if $f$ is cts at $x$ as $r \to 1$.

② Whenever $f$ is cts on the circle,

$$P_r * f \rightrightarrows f \quad \text{as } r \to 1.$$

Recall that

$$P_r * f(x) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{inx}$$

$$( := A_r f(x) \quad \text{—— Abel mean of the Fourier series of } f)$$

## § 2.5 Applications to the steady-state heat equation on the unit disc.

Consider the heat distribution on a (very thin) metal plate.

$u(x, y, t)$ —— the temperature at the point $(x, y)$ at time $t$.

Then $u$ satisfies the following

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x, y) \in \mathbb{R}^2, \ t > 0.$$

In the special case when $u$ is independent of $t$,

then $u = u(x,y)$ satisfies

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad (x,y) \in \mathbb{R}^2.$$

(Steady-state heat equation)

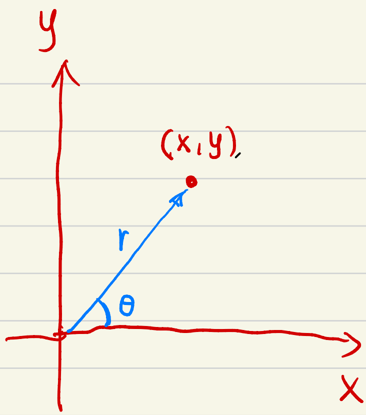
Now consider the unit disc

$$D := \{ (x,y) : x^2 + y^2 < 1 \}.$$

We want to consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x,y) \in D.$$

We would like to re-express $D$ and the heat equation in the polar coordinates:




$(r, \theta)$, with $r \geq 0$, $\theta \in [0, 2\pi)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Fig. Polar coordinate $(r, \theta)$ for the point $(x, y) \in \mathbb{R}^2$.

Using the polar coordinate, the unit disc can be expressed as

$$D = \{ (r, \theta) : 0 \leq r < 1, \ 0 \leq \theta < 2\pi \}$$

Moreover, the steady-state heat equation can be rewritten as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Thm 1. Let $f$ be integrable on the circle.

Let $u(r,\theta) := P_r * f(\theta)$, $0 \le r < 1$, $0 \le \theta < 2\pi$.

Then (1) $u \in C^2(D)$. Moreover

$$\Delta u := \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

(Laplace operator)

(2) If $f$ is cts at $\theta$, then

$$\lim_{r \to 1} u(r,\theta) = f(\theta).$$

Recall :

Let $J$ be an interval on $\mathbb{R}$.

Suppose $(f_n)$ is a sequence of cts functions on $J$ such that

① $f_n(x_0) \longrightarrow f(x_0)$

② $f_n' \rightrightarrows g$ on $J$

Then $\exists f$ s.t

$f_n \rightrightarrows f$ on $J$

and $f' = g$.

As a special consequence, if

$$\sum_{n=1}^{N} S_n(x) \rightarrow S(x) \quad \text{for all } x \in J$$

and

$$\sum_{n=1}^{N} S_n'(x) \rightrightarrows g(x) \quad \text{on } J,$$

as $N \to \infty$. Then $S'(x) = g(x)$ on $J$, i.e.

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} S_n(x)\right) = \sum_{n=1}^{\infty} \frac{d}{dx} S_n(x) \quad \text{on } J.$$

Pf of Thm 1 (i) :

We first show $u \in C^2(D)$.

Recall that

$$P_r * f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{in\theta},$$

$0 \le r < 1$, $0 \le \theta < 2\pi$.

Notice that the above series converges uniformly on the region

$$\{ (r, \theta) ; 0 \le r < \rho \}$$

for any $0 < \rho < 1$.

Also,
$$\sum_{n=-\infty}^{\infty} \frac{\partial}{\partial r} \left( r^{|n|} \hat{f}(n) e^{in\theta} \right)$$

$$\sum_{n=-\infty}^{\infty} \frac{\partial}{\partial \theta} \left( r^{|n|} \hat{f}(n) e^{in\theta} \right)$$

Converge uniformly on $\{ (r, \theta) ; 0 \le r < \rho \}$

Hence $\frac{\partial}{\partial r}(P_r * f)$

$$= \sum_{n=-\infty}^{\infty} \frac{\partial}{\partial r}\left( r^{|n|} \hat{f}(n) e^{in\theta} \right)$$

$\frac{\partial}{\partial \theta}(P_r * f)$

$$= \sum_{n=-\infty}^{\infty} \frac{\partial}{\partial \theta}\left( r^{|n|} \hat{f}(n) e^{in\theta} \right)$$

which means that $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$ exist on D. Similarly, we can that $u$ is $C^\infty$ on D

Next we check $\Delta u = 0$.

Notice that

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \sum_{n=-\infty}^{\infty} \left[ \frac{\partial^2}{\partial r^2} \left( r^{|n|} \hat{f}(n) e^{in\theta} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r^{|n|} \hat{f}(n) e^{in\theta} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \left( r^{|n|} \hat{f}(n) e^{in\theta} \right) \right]$$

So we only need to check that for given $n \in \mathbb{Z}$,

$$\Delta \left( r^{|n|} e^{in\theta} \right) = 0.$$

Let us check it in the case when $n=3$.

$$\frac{\partial^2}{\partial r^2} \left( r^3 e^{i \cdot 3\theta} \right) = 6 r e^{i 3\theta}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r^3 e^{i3\theta} \right) = 3 r e^{i3\theta}$$

$$\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \left( r^3 e^{i3\theta} \right) = r \cdot (3i)^2 e^{i3\theta} = -9 r e^{i3\theta}$$

Hence

$$\Delta\left( r^3 e^{i3\theta} \right) = 0.$$

$\blacksquare$